A PROBLEM OF OPTIMUM CONTROL

(ODNA ZADACHA OPTIMAL'NOGO UPRAVLENIIA)

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1. We shall investigate the control system (Fig.1)

$$x^{\cdot\cdot} = u(t), \qquad x(0) = x^{\cdot}(0) = 0 \tag{1.1}$$

$$0 \leqslant u(t) \leqslant u_2 \quad (0 \leqslant t \leqslant T), \qquad u_1 \leqslant u(t) \leqslant u_2 \quad (t > T) \tag{1.2}$$

where the time $t = T$ is fixed. We shall call an arbitrary, piece-wise continuous function $u(t)$ satisfying the restraints (1.2) which has a finite number of discontinuities of the first kind on any interval $t_1 \leqslant t \leqslant t_2$, an admissible control. The following variational problem now arises: to find the control signal $u = u_*(t)$ which is a member of the class of admissible controls and which ensures control action on the variable at $x = x_*$ with minimal velocity. The control $u = u_*(t)$ we shall call the optimum control.

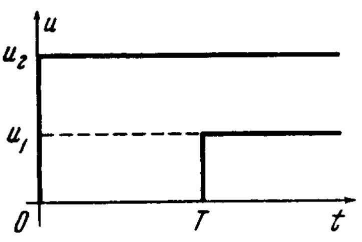

Fig. 1

Let us introduce a function

$$\varphi(t) = \int_0^t u(t)\, dt \tag{1.3}$$

It can be seen that by (1.2), the maximum ordinate $\varphi(t)$ over an arbitrary interval $t_1 \leqslant t \leqslant t_2$ is

$$\max_t \varphi(t) = \varphi(t_2) \qquad (t_1 \leqslant t \leqslant t_2) \tag{1.4}$$

Hence, the above problem can be reformulated as follows: out of all curves $\varphi(t)$ possessing $\varphi^{\cdot}(t)$, the latter satisfying the relations

$$0 \leqslant \varphi^{\cdot}(t) \leqslant u_2 \quad (0 \leqslant t \leqslant T), \qquad u_1 \leqslant \varphi^{\cdot}(t) \leqslant u_2 \quad (t > T) \tag{1.5}$$

to find a curve $\varphi = \varphi_*(t)$ the ordinate of which will, at the instant $t = t_*$ given by the condition

$$\int_0^{t_*} \varphi_*(t)\, dt = x_* \tag{1.6}$$

attain a minimum.

Let us now denote by Φ a set of curves (1.3) satisfying conditions (1.5) and (1.6). Let Ψ be a subset of Φ, the subset composed of continuous lines $\psi(\tau, t)$ with different discrete slopes over particular intervals, each line dependent on the parameter τ

$$\psi(\tau, t) = \begin{cases} u_2 t & (0 \leqslant t \leqslant \tau) \\ u_2\tau & (\tau \leqslant t \leqslant T) \\ u_2\tau + u_1(t - T) & (T \leqslant t \leqslant t_*) \end{cases} \tag{1.7}$$

where for any $\tau < T$, the instant $t = t_*$ is determined from

$$\int_0^{t_*} \psi(\tau, t)\, dt = x_* \tag{1.8}$$

We shall show that if the condition

$$t_* \geqslant T \tag{1.9}$$

is satisfied, then the function

$$\varphi_*(t) = \int_0^t u_*(t)\, dt \qquad (0 \leqslant t \leqslant t_*) \tag{1.10}$$

where $u_*(t)$ is the optimum control signal anf the time $t = t_*$ is determined by (1.6), belongs to the set Ψ. Indeed, let some curve $\varphi^\circ(t)$, $0 \leqslant t \leqslant t^\circ \geqslant T$, where

$$\int_0^{t^\circ} \varphi^\circ(t)\, dt = x_* \tag{1.11}$$

which is a solution of our problem, be not a member of the set Ψ . Let us define a point on this curve by the value of its abscissa, namely $t = t^\circ$, and let us draw through it a line $\psi(\tau_1, t)$, the parameter τ_1 of which is uniquely determinable (Fig.2). If, at the same time $\varphi^\circ(t) \not\equiv \psi(\tau_1, t)$, $0 \leqslant t \leqslant t^\circ$, which we shall assume from now on, then, by (1.5) and (1.7), the relations

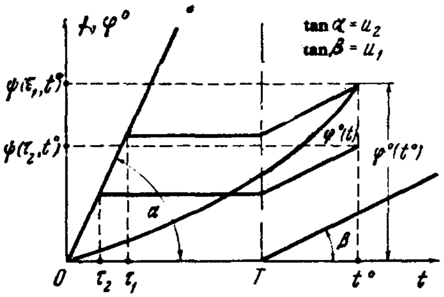

Fig. 2

$$\psi(\tau_1, t) \geqslant \varphi^\circ(t) \qquad (0 \leqslant t \leqslant t^\circ),$$

$$\int_0^{t^\circ} \psi(\tau_1, t)\, dt > x_* \tag{1.12}$$

will obviously be fulfilled for the ordinates of the curves in question.

Hence, we should be able to construct a line $\psi(\tau_2, t), 0 \leqslant t \leqslant t^\circ$, with the parameter $\tau_2 < \tau_1$ given by

$$\int_0^{t^\circ} \psi(\tau_2, t)\, dt = x_* \tag{1.13}$$

At $t = t^\circ$, the value of the ordinate of this line will be less than $\varphi^\circ(t^\circ)$, which contradicts the assumption that $\varphi^\circ(t)$ is optimum. In this manner we have reduced the initial variational problem to the problem of finding the minimum (still assuming that the inequality (1.9) is fulfilled), of the function

$$\psi(\tau, t_*) = u_2\tau + u_1(t_* - T) \tag{1.14}$$

the variables τ and t_* of which, satisfy the relation

$$\chi(\tau, t_*) = u_2\tau t_* - {}^1/_2 u_2\tau^2 + {}^1/_2 u_1(t_* - T)^2 - x_* = 0 \tag{1.15}$$

The unknowns τ and t_* are found from

$$\partial F / \partial\tau = 0, \qquad \partial F / \partial t_* \tag{1.16}$$

together with (1.15), where

$$F(\tau, t_*, \lambda) = \psi(\tau, t_*) + \lambda\chi(\tau, t_*) \tag{1.17}$$

where λ is a multiplier. After the necessary transformations, we obtain

$$\tau = \frac{u_1}{u_1 + u_2} T, \qquad t_* = \frac{u_1}{u_1 + u_2} T + \left[\frac{2x_*}{u_1} - \frac{u_2}{u_1 + u_2} T^2\right]^{1/2} \tag{1.18}$$

from which it follows that (1.9) can occur if and only if

$$\left[\frac{2x_*}{u_1} - \frac{u_2}{u_1+u_2}T^2\right]^{1/2} \geqslant \frac{u_2}{u_1+u_2}T \tag{1.19}$$

which yields

$$x_* \geqslant \frac{1+2u_2/u_1}{(1+u_2/u_1)^2}\frac{u_2T^2}{2} \tag{1.20}$$

When (1.20) is fulfilled, then the optimum control signal $u_*(t)$ is governed by the following law:

$$u_*(t) = \begin{cases} u_2 & (0 \leqslant t \leqslant \tau) \\ 0 & (\tau < t \leqslant T) \\ u_1 & (T < t \leqslant t_*) \end{cases} \tag{1.21}$$

where the values of τ and t_* are determined from (1.18). The velocity of the action of the coordinate x_* is

$$\min_u \max_t x^{\cdot}(t) = \psi(\tau, t_*) = u_1\left[\frac{2x_*}{u_1} - \frac{u_2}{u_1+u_2}T^2\right]^{1/2} \tag{1.22}$$

Next we shall show the necessity of fulfilling the condition (1.9) for the function (1.10). Indeed, let us assume that, for the curve $\varphi_0(t)$, $0 \leqslant t \leqslant t_0$ representing the solution of the initial problem, the instant $t = t_0$ found by means of Formula

$$\int_0^{t_0} \varphi_0(t)\,dt = x_* \tag{1.23}$$

satisfies the inequality $t_0 < T$. We shall now denote by B (Fig.3) a point on the curve $\varphi_0(t)$ corresponding to $t = t_0$ and draw through B a line

$$\psi_0(\tau, t) = \begin{cases} u_2 t & (0 \leqslant t \leqslant \tau) \\ u_2\tau & (\tau \leqslant t \leqslant t_0) \end{cases} \tag{1.24}$$

the parameter $\tau = \tau_1$ of which is uniquely determinable. If at the same time $\psi_0(\tau_1, t) \not\equiv \varphi_0(t)$, then by (1.5) and (1.24), the relations (1.12) in which $\psi(\tau_1, t), \varphi^0(t)$ and t^0 should be replaced by $\psi_0(\tau_1, t), \varphi_0(t)$ and t_0, respectively, are fulfilled. Consequently, we can construct a curve $\psi_0(\tau_2, t)$ with the parameter $\tau_2 < \tau_1$ satisfying (1.13) (where $\psi(\tau_2, t)$ and t^0 are replaced by $\psi_0(\tau_2, t)$ and t_0). Its ordinate at $t = t_0$ is smaller than $\psi_0(\tau_1, t_0)$. Moreover, in this case (as well as in case when $\psi_0(\tau_1, t) \equiv \varphi_0(t)$, $0 \leqslant t \leqslant t_0$) we can construct a line $\psi_0(\tau_0, t)$, the parameter $\tau_0 < \tau_2$ of which can be found from

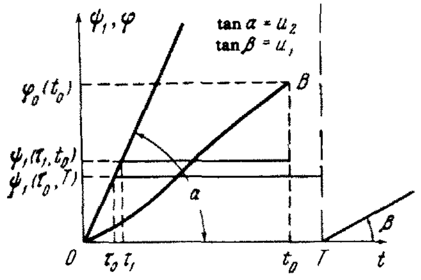

Fig. 3

$$\int_0^T \psi_0(\tau_0, t)\,dt = x_* \tag{1.25}$$

From this it follows that the maximum value of the ordinate

$$\psi_0(\tau_0, T) = u_2\tau_0 \tag{1.26}$$

of the curve $\psi_0(\tau_0, t)$ is lower than the values of $\varphi_0(t_0)$ and $\psi_0(\tau_2, t)$, which contradicts the optimum condition for $\varphi_0(t)$ and at the same time proves the necessity of satisfying the inequality (1.9). The above gives us also a method for constructing the optimum control in case, when

$$x_* \leqslant \frac{1 + 2u_2 / u_1}{(1 + u_2 / u_1)^2} \frac{u_2 T^2}{2} \tag{1.27}$$

i.e. when Equations (1.18) contradict (1.9). When the relation (1.27) is fulfilled, the optimum control signal $u = u_*(t)$ is determined by the function $\psi_0(\tau_0, t)$, $0 \leqslant t \leqslant T$, and has the form

$$u_*(t) = u_2 \quad (0 \leqslant t \leqslant \tau_0), \qquad u_*(t) = 0 \quad (\tau_0 < t \leqslant T) \tag{1.28}$$

where the instant τ_0 of the switch-over is found from

$$\tau_0 = T - [T^2 - 2x_* / u_2)]^{1/2} \tag{1.29}$$

obtained by substituting $\psi_0(\tau_0, t)$ into (1.25).

The control action velocity on x_* is

$$\min_u \max_t x^\circ(t) = u_2 \tau_0 = u_2 \{T - [T^2 - 2x_* / u_2]^{1/2}\} \tag{1.30}$$

and the action occurs at the instant $t = T$. It can be directly verified that on

$$x_* = \frac{1 + 2u_2 / u_1}{(1 + u_2 / u_1)^2} \frac{u_2 T^2}{2} \tag{1.31}$$

the switch-over instants and velocities of control actions on x_* obtained by means of (1.18), (1.29) and of (1.22), (1.30), are in full agreement.

2. We shall now consider a solution of our problem with another condition added : that the optimum control will be maintained when $t \geqslant 0$. We shall introduce the set Ψ_1 (Fig.3) of continuous lines $\psi_1(U, t)$, $0 \leqslant t \leqslant t_*$, with a parameter U

$$\psi_1(U, t) = \begin{cases} Ut & (0 \leqslant t \leqslant T) \\ UT + u_1(t - T) & (T \leqslant t \leqslant t_*) \end{cases} \tag{2.1}$$

where $0 \leqslant U \leqslant u_2$, and the time t_* is found from

$$\int_0^{t_*} \psi_1(U, t)\, dt = x_* \tag{2.2}$$

Utilizing our previous arguments in Section 1, we shall show that when

$$t_* \geqslant T \tag{2.3}$$

holds, then the function (1.10) is a member of the set Ψ_1, so that the initial variational problem is reduced to finding the minimum of the function

$$\psi_1(U, t_*) = UT + u_1(t_* - T) \tag{2.4}$$

with the functional relation between the variables U and t_* given by

$$\chi_1(U, t_*) = UTt_* - {}^1/_2 UT^2 + {}^1/_2 u_1 (t_* - T)^2 - x_* = 0 \tag{2.5}$$

which in turn follows from (2.2). Omitting the intermediate reasoning, we arrive at the values of U and t_* for which (2.4) is at minimum

$$U = {}^1/_2 u_1, \qquad t_* = {}^1/_2 T + [2x_* / u_1 - {}^1/_4 T^2]^{1/2} \tag{2.6}$$

Here the optimum control is

$$u_*(t) = {}^1/_2 u_1 \quad (0 \leqslant t \leqslant T), \qquad u_*(t) = u_1 \quad (T < t \leqslant t_*) \tag{2.7}$$

and the velocity with which it acts on x_* at $t = t_*$ is

$$\min_u \max_t x^\cdot(t) = UT + u_1(t_* - T) = u_1[2x_* / u_1 - {}^1/_4 T^2]^{1/2} \tag{2.8}$$

From the above formulas it follows that (2.3) holds if and only if

$$[2x_* / u_1 - \tfrac{1}{4}T^2]^{1/2} \geqslant \tfrac{1}{2}T \quad \text{, or} \quad x_* \geqslant \tfrac{1}{4}u_1 T^2 \tag{2.9}$$

Proof can be obtained as in the analogous case in the Section 1. Hence, when

$$x_* \leqslant \tfrac{1}{4}u_1 T^2 \tag{2.10}$$

the optimum control $u_*(t)$ will become

$$u_*(t) = U_0 \qquad (0 \leqslant t \leqslant T), \qquad U_0 = 2\,(x_* / T^2) \tag{2.11}$$

where U_0 is found from

$$\int_0^T U_0 t \, dt = x_* \tag{2.12}$$

Velocity of control action on x_* at $t = T$ becomes

$$\min_u \max_t x^{\cdot}(t) = U_0 T = 2x_* / T \tag{2.13}$$

and we can verify directly that when the condition

$$x_* = \tfrac{1}{4}u_1 T^2$$

is fulfilled, the parameters U and U_0 of the optimum controls and velocities of control action on x_* as calculated by means of (2.6),(2.11),(2.8) and (2.13), are in full agreement.

Translated by L.K.